



Hypo- q -norms on cartesian products of algebras of bounded linear operators on Hilbert spaces

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Abstract: In this paper we introduce the hypo- q -norms on a Cartesian product of algebras of bounded linear operators on Hilbert spaces. A representation of these norms in terms of inner products, the equivalence with the q -norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given. Several bounds for the norms δ_p , ϑ_p and the real norms $\eta_{r,p}$ and $\theta_{r,p}$ are provided as well.

Key words: Hilbert spaces, bounded linear operators, operator norm and numerical radius, n -tuple of operators, operator inequalities.

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1. INTRODUCTION

In [13], the author has introduced the following norm on the Cartesian product $B^{(n)}(H) := B(H) \times \cdots \times B(H)$, where $B(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1, \dots, T_n)\|_{n,e} := \sup_{(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \quad (1.1)$$

where $(T_1, \dots, T_n) \in B^{(n)}(H)$ and

$$\mathbb{B}_n := \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right\}$$

is the Euclidean closed ball in \mathbb{C}^n . It is clear that $\|\cdot\|_{n,e}$ is a norm on $B^{(n)}(H)$ and for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ we have

$$\|(T_1, \dots, T_n)\|_{n,e} = \|(T_1^*, \dots, T_n^*)\|_{n,e}, \quad (1.2)$$



where T_i^* is the adjoint operator of T_i , $i \in \{1, \dots, n\}$.

It has been shown in [13] that the following inequality holds true:

$$\frac{1}{\sqrt{n}} \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \leq \|(T_1, \dots, T_n)\|_{n,e} \leq \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \quad (1.3)$$

for any n -tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [13] the author has introduced the *Euclidean operator radius* of an n -tuple of operators (T_1, \dots, T_n) by

$$w_{n,e}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \quad (1.4)$$

and proved that $w_{n,e}(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2} \|(T_1, \dots, T_n)\|_{n,e} \leq w_{n,e}(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_{n,e} \quad (1.5)$$

for each n -tuple $(T_1, \dots, T_n) \in B^{(n)}(H)$.

As pointed out in [13], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}} \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \leq w_{n,e}(T_1, \dots, T_n) \leq \left\| \sum_{j=1}^n T_j T_j^* \right\|^{\frac{1}{2}} \quad (1.6)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

Now, let $(E, \|\cdot\|)$ be a normed linear space over the complex number field \mathbb{C} . On \mathbb{C}^n endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$. As an example of such norms we should mention the usual p -norms

$$\|\lambda\|_{n,p} := \begin{cases} \max \{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty, \\ \left(\sum_{k=1}^n |\lambda_k|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\lambda\|_{n,2} := \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following p -norms:

$$\|x\|_{n,p} := \begin{cases} \max \{|x_1|, \dots, |x_n|\} & \text{if } p = \infty, \\ \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases}$$

where $x = (x_1, \dots, x_n) \in E^n$.

Following the paper [5], for a given norm $\|\cdot\|_n$ on \mathbb{C}^n , we define the functional $\|\cdot\|_{h,n} : E^n \rightarrow [0, \infty)$ by

$$\|x\|_{h,n} := \sup_{\|\lambda\|_n \leq 1} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \quad (1.7)$$

where $x = (x_1, \dots, x_n) \in E^n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

It is easy to see that [5]:

- (i) $\|x\|_{h,n} \geq 0$ for any $x \in E^n$,
- (ii) $\|x + y\|_{h,n} \leq \|x\|_{h,n} + \|y\|_{h,n}$ for any $x, y \in E^n$,
- (iii) $\|\alpha x\|_{h,n} = |\alpha| \|x\|_{h,n}$ for each $\alpha \in \mathbb{C}$ and $x \in E^n$,

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n . This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n .

We observe that $\|x\|_{h,n} = 0$ if and only if $\sum_{j=1}^n \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \dots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \dots, 0)$, $(0, \lambda_2^0, \dots, 0)$, \dots , $(0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n .

If $p \in [1, \infty]$ and we consider the p -norms $\|\cdot\|_{n,p}$ on \mathbb{C}^n , then we can define the following *hypo- q -norms* on E^n :

$$\|x\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \quad (1.8)$$

with $q \in [1, \infty]$. If $p = 1$, then $q = \infty$; if $p = \infty$, then $q = 1$; if $p \in (1, \infty)$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$, we have the *hypo-Euclidean norm* on E^n , i.e.,

$$\|x\|_{h,n,e} := \sup_{\|\lambda\|_{n,2} \leq 1} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \quad (1.9)$$

If we consider now $E = B(H)$ endowed with the operator norm $\|\cdot\|$, then we can obtain the following *hypo- q -norms* on $B^{(n)}(H)$

$$\|(T_1, \dots, T_n)\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \quad \text{where } p, q \in [1, \infty], \quad (1.10)$$

with the convention that if $p = 1$, $q = \infty$, if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$ we obtain the *hypo-Euclidian norm* $\|(\cdot, \dots, \cdot)\|_{n,e}$ defined in (1.2).

If we consider now $E = B(H)$ endowed with the operator numerical radius $w(\cdot)$, which is a norm on $B(H)$, then we can obtain the following *hypo- q -numerical radius* of $(T_1, \dots, T_n) \in B^{(n)}(H)$ defined by

$$w_{h,n,q}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,p} \leq 1} w\left(\sum_{j=1}^n \lambda_j T_j\right) \quad \text{with } p, q \in [1, \infty], \quad (1.11)$$

with the convention that if $p = 1$, $q = \infty$, if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$ we obtain the *hypo-Euclidian norm*

$$w_{h,n,e}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,2} \leq 1} w\left(\sum_{j=1}^n \lambda_j T_j\right) \quad (1.12)$$

and will show further that it coincides with the Euclidean operator radius of an n -tuple of operators (T_1, \dots, T_n) defined in (1.4).

Using the fundamental inequality between the operator norm and numerical radius $w(T) \leq \|T\| \leq 2w(T)$ for $T \in B(H)$ we have

$$w\left(\sum_{j=1}^n \lambda_j T_j\right) \leq \left\| \sum_{j=1}^n \lambda_j T_j \right\| \leq 2w\left(\sum_{j=1}^n \lambda_j T_j\right)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. By taking the supremum over λ with $\|\lambda\|_{n,p} \leq 1$ we get

$$w_{h,n,q}(T_1, \dots, T_n) \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq 2w_{h,n,q}(T_1, \dots, T_n) \quad (1.13)$$

with the convention that if $p = 1$, $q = \infty$, if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = q = 2$ we recapture the inequality (1.5).

In 2012, [8] (see also [9, 10]) the author have introduced the concept of s - q -numerical radius of an n -tuple of operators (T_1, \dots, T_n) for $q \geq 1$ as

$$w_{s,q}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} \quad (1.14)$$

and established various inequalities of interest. For more recent results see also [12, 14].

In the same paper [8] we also introduced the concept of s - q -norm of an n -tuple of operators (T_1, \dots, T_n) for $q \geq 1$ as

$$\|(T_1, \dots, T_n)\|_{s,q} := \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q}. \quad (1.15)$$

In [8], [9] and [10], by utilising *Kato's inequality* [11]

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle \quad (1.16)$$

for any $x, y \in H$, $\alpha \in [0, 1]$, where “absolute value” operator of A is defined by $\|A\| := \sqrt{A^*A}$, the authors have obtained several inequalities for the s - q -numerical radius and s - q -norm.

In this paper we investigate the connections between these norms and establish some fundamental inequalities of interest in multivariate operator theory.

2. REPRESENTATION RESULTS

We start with the following lemma:

LEMMA 1. Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$.

(i) If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,q}. \quad (2.1)$$

In particular,

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,2}. \quad (2.2)$$

(ii) We have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,1} \quad \text{and} \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_{n,\infty}. \quad (2.3)$$

Proof. (i) Using Hölder's discrete inequality for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \left(\sum_{j=1}^n |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^n |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,q} \quad (2.4)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are n -tuples of complex numbers.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with

$$\alpha_j := \frac{\overline{\beta_j} |\beta_j|^{q-2}}{\left(\sum_{k=1}^n |\beta_k|^q \right)^{1/p}}$$

for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We observe that

$$\begin{aligned} \left| \sum_{j=1}^n \alpha_j \beta_j \right| &= \left| \sum_{j=1}^n \frac{\overline{\beta_j} |\beta_j|^{q-2}}{\left(\sum_{k=1}^n |\beta_k|^q \right)^{1/p}} \beta_j \right| = \frac{\sum_{j=1}^n |\beta_j|^q}{\left(\sum_{k=1}^n |\beta_k|^q \right)^{1/p}} \\ &= \left(\sum_{j=1}^n |\beta_j|^q \right)^{1/q} = \|\beta\|_{n,q} \end{aligned}$$

and

$$\begin{aligned} \|\alpha\|_{n,p}^p &= \sum_{j=1}^n |\alpha_j|^p = \sum_{j=1}^n \frac{\left| \overline{\beta_j} |\beta_j|^{q-2} \right|^p}{\left(\sum_{k=1}^n |\beta_k|^q \right)} = \sum_{j=1}^n \frac{\left(|\beta_j|^{q-1} \right)^p}{\left(\sum_{k=1}^n |\beta_k|^q \right)} \\ &= \sum_{j=1}^n \frac{|\beta_j|^{qp-p}}{\left(\sum_{k=1}^n |\beta_k|^q \right)} = \sum_{j=1}^n \frac{|\beta_j|^q}{\left(\sum_{k=1}^n |\beta_k|^q \right)} = 1. \end{aligned}$$

Therefore, by (2.4) we have the representation (2.1).

(ii) Using the properties of the modulus, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \max_{j \in \{1, \dots, n\}} |\alpha_j| \sum_{j=1}^n |\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,1}, \quad (2.5)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \sum_{j=1}^n \frac{\overline{\beta_j}}{|\beta_j|} \beta_j \right| = \sum_{j=1}^n |\beta_j| = \|\beta\|_{n,1}$$

and

$$\|\alpha\|_{n,\infty} = \max_{j \in \{1, \dots, n\}} |\alpha_j| = \max_{j \in \{1, \dots, n\}} \left| \frac{\overline{\beta_j}}{|\beta_j|} \right| = 1$$

and by (2.5) we get the first representation in (2.3).

Moreover, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \sum_{j=1}^n |\alpha_j| \max_{j \in \{1, \dots, n\}} |\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_{n,\infty}, \quad (2.6)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, let $j_0 \in \{1, \dots, n\}$ such that

$$\|\beta\|_{\infty} = \max_{j \in \{1, \dots, n\}} |\beta_j| = |\beta_{j_0}|.$$

Consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_{j_0} = \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j = 0$ for $j \neq j_0$. For this choice we get

$$\sum_{j=1}^n |\alpha_j| = \frac{|\overline{\beta_{j_0}}|}{|\beta_{j_0}|} = 1 \quad \text{and} \quad \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|} \beta_{j_0} \right| = |\beta_{j_0}| = \|\beta\|_{n,\infty},$$

therefore by (2.6) we obtain the second representation in (4). ■

THEOREM 2. *Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $x, y \in H$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \quad (2.7)$$

and in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{1/2}. \quad (2.8)$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \sum_{j=1}^n |\langle T_j x, y \rangle| \quad (2.9)$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| = \max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\}. \quad (2.10)$$

Proof. If we take $\beta = (\langle T_1 x, y \rangle, \dots, \langle T_n x, y \rangle) \in \mathbb{C}^n$ in (2.1), then we get

$$\begin{aligned} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} &= \|\beta\|_{n,q} = \sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^n \alpha_j \langle T_j x, y \rangle \right| = \sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \sum_{j=1}^n \alpha_j T_j x, y \right\rangle \right|, \end{aligned}$$

which proves (2.7).

The equalities (2.9) and (2.10) follow by (2.3). ■

COROLLARY 3. *Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $x \in H$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} \quad (2.11)$$

and, in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{1/2}. \quad (2.12)$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \sum_{j=1}^n |\langle T_j x, x \rangle| \quad (2.13)$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| = \max_{j \in \{1, \dots, n\}} \{|\langle T_j x, x \rangle|\}. \quad (2.14)$$

COROLLARY 4. Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $x \in H$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \quad (2.15)$$

and in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{1/2}. \quad (2.16)$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \sup_{\|y\|=1} \sum_{j=1}^n |\langle T_j x, y \rangle| \quad (2.17)$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| = \max_{j \in \{1, \dots, n\}} \{\|T_j x\|\}. \quad (2.18)$$

Proof. By the properties of inner product, we have for any $u \in H$, $u \neq 0$ that

$$\|u\| = \sup_{\|y\|=1} |\langle u, y \rangle|.$$

Let $x \in H$, then by taking the supremum over $\|y\| = 1$ in (2.7) we get for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned} \sup_{\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|y\|=1} \left(\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left(\sup_{\|y\|=1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, y \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \left(\sum_{j=1}^n \alpha_j T_j \right) x \right\|, \end{aligned}$$

which proves the equality (2.15).

The other equalities can be proved in a similar way by using Theorem 2, however the details are omitted. ■

We can state and prove our main result.

THEOREM 5. *Let $(T_1, \dots, T_n) \in B^{(n)}(H)$.*

(i) *For $q \geq 1$ we have the representation for the hypo- q -norm*

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{h,n,q} &= \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \\ &= \|(T_1, \dots, T_n)\|_{s,q} \end{aligned} \quad (2.19)$$

and in particular

$$\|(T_1, \dots, T_n)\|_{n,e} = \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{1/2}. \quad (2.20)$$

We also have

$$\|(T_1, \dots, T_n)\|_{h,n,\infty} = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}. \quad (2.21)$$

(ii) For $q \geq 1$ we have the representation for the hypo-numerical radius

$$\begin{aligned} w_{h,n,q}(T_1, \dots, T_n) &= \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} \\ &= w_{s,q}(T_1, \dots, T_n) \end{aligned} \quad (2.22)$$

and in particular

$$w_{n,e}(T_1, \dots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{1/2}. \quad (2.23)$$

We also have

$$w_{h,n,\infty}(T_1, \dots, T_n) = \max_{j \in \{1, \dots, n\}} \{w(T_j)\}. \quad (2.24)$$

Proof. (i) By using the equality (2.15) we have for $(T_1, \dots, T_n) \in B^{(n)}(H)$ that

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|y\|=1} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \right) \\ &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left(\sup_{\|x\|=1} \left\| \sum_{j=1}^n \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \alpha_j T_j \right\| \\ &= \|(T_1, \dots, T_n)\|_{h,n,q}, \end{aligned}$$

which proves (2.19). The rest is obvious.

(ii) By using the equality (2.11) we have for $(T_1, \dots, T_n) \in B^{(n)}(H)$ that

$$\begin{aligned} \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} \left(\sup_{\|x\|=1} \left| \left\langle \left(\sum_{j=1}^n \alpha_j T_j \right) x, x \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \leq 1} w \left(\sum_{j=1}^n \alpha_j T_j \right) = w_{h,n,q}(T_1, \dots, T_n), \end{aligned}$$

which proves (2.22). The rest is obvious. ■

Remark 6. The case $q = 2$ was obtained in a different manner in [5] by utilising the rotation-invariant normalised positive Borel measure on the unit sphere.

We can consider on $B^{(n)}(H)$ the following usual operator and numerical radius q -norms, for $q \geq 1$

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,q} &:= \left(\sum_{j=1}^n \|T_j\|^q \right)^{1/q}, \\ w_{n,q}(T_1, \dots, T_n) &:= \left(\sum_{j=1}^n w^q(T_j) \right)^{1/q}, \end{aligned}$$

where $(T_1, \dots, T_n) \in B^{(n)}(H)$. For $q = \infty$ we put

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,\infty} &:= \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}, \\ w_{n,\infty}(T_1, \dots, T_n) &:= \max_{j \in \{1, \dots, n\}} \{w(T_j)\}. \end{aligned}$$

COROLLARY 7. *With the assumptions of Theorem 5 we have for $q \geq 1$ that*

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{n,q} \quad (2.25)$$

and

$$\frac{1}{n^{1/q}} w_{n,q}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq w_{n,q}(T_1, \dots, T_n) \quad (2.26)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

In particular, we have [5]

$$\frac{1}{\sqrt{n}} \|(T_1, \dots, T_n)\|_{n,2} \leq \|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{n,2} \quad (2.27)$$

and

$$\frac{1}{\sqrt{n}} w_{n,2}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq w_{n,2}(T_1, \dots, T_n) \quad (2.28)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Proof. Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $x, y \in H$ with $\|x\| = \|y\| = 1$. Then by Schwarz's inequality we have

$$\left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \leq \left(\sum_{j=1}^n \|T_j x\|^q \|y\|^q \right)^{1/q} = \left(\sum_{j=1}^n \|T_j x\|^q \right)^{1/q}.$$

By the operator norm inequality we also have

$$\left(\sum_{j=1}^n \|T_j x\|^q \right)^{1/q} \leq \left(\sum_{j=1}^n \|T_j\|^q \|x\|^q \right)^{1/q} = \|(T_1, \dots, T_n)\|_{n,q}.$$

Therefore

$$\left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \leq \|(T_1, \dots, T_n)\|_{n,q}$$

and by taking the supremum over $\|x\| = \|y\| = 1$ we get the second inequality in (2.25).

By the properties of complex numbers, we have

$$\max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\} \leq \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q}$$

$x, y \in H$ with $\|x\| = \|y\| = 1$.

By taking the supremum over $\|x\| = \|y\| = 1$ we get

$$\sup_{\|x\|=\|y\|=1} \left(\max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\} \right) \leq \|(T_1, \dots, T_n)\|_{h,n,q} \quad (2.29)$$

and since

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left(\max_{j \in \{1, \dots, n\}} \{|\langle T_j x, y \rangle|\} \right) &= \max_{j \in \{1, \dots, n\}} \left\{ \sup_{\|x\|=\|y\|=1} |\langle T_j x, y \rangle| \right\} \\ &= \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} = \|(T_1, \dots, T_n)\|_{n,\infty}, \end{aligned}$$

then by (2.29) we get

$$\|(T_1, \dots, T_n)\|_{n,\infty} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \quad (2.30)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Since

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{n,q} &:= \left(\sum_{j=1}^n \|T_j\|^q \right)^{1/q} \leq (n \|(T_1, \dots, T_n)\|_{n,\infty}^q)^{1/q} \\ &= n^{1/q} \|(T_1, \dots, T_n)\|_{n,\infty}, \end{aligned} \quad (2.31)$$

then by (2.30) and (2.31) we get

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,q}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

The inequality (2.26) follows in a similar way and we omit the details. ■

COROLLARY 8. *With the assumptions of Theorem 5 we have for $r \geq q \geq 1$ that*

$$\|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|(T_1, \dots, T_n)\|_{h,n,r} \quad (2.32)$$

and [14]

$$w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq n^{\frac{r-q}{rq}} w_{h,n,r}(T_1, \dots, T_n) \quad (2.33)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j , $j = 1, \dots, n$ and $r \geq q > 0$ (see for instance [14])

$$\left(\sum_{j=1}^n a_j^r \right)^{1/r} \leq \left(\sum_{j=1}^n a_j^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^n a_j^r \right)^{1/r}. \quad (2.34)$$

Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $x, y \in H$ with $\|x\| = \|y\| = 1$. Then by (2.34) we get

$$\left(\sum_{j=1}^n |\langle T_j x, y \rangle|^r \right)^{1/r} \leq \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^n |\langle T_j x, y \rangle|^r \right)^{1/r}.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get (2.32).

The inequality (2.33) follows in a similar way and we omit the details. ■

Remark 9. For $q \geq 2$ we have by (2.32) and (2.33)

$$\|(T_1, \dots, T_n)\|_{h,n,q} \leq \|(T_1, \dots, T_n)\|_{h,n,e} \leq n^{\frac{q-2}{2q}} \|(T_1, \dots, T_n)\|_{h,n,q} \quad (2.35)$$

and

$$w_{h,n,q}(T_1, \dots, T_n) \leq w_{h,n,e}(T_1, \dots, T_n) \leq n^{\frac{q-2}{2q}} w_{h,n,q}(T_1, \dots, T_n) \quad (2.36)$$

and for $1 \leq q \leq 2$ we have

$$\|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|(T_1, \dots, T_n)\|_{h,n,e} \quad (2.37)$$

and

$$w_{h,n,e}(T_1, \dots, T_n) \leq w_{h,n,q}(T_1, \dots, T_n) \leq n^{\frac{2-q}{2q}} w_{h,n,e}(T_1, \dots, T_n) \quad (2.38)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

Also, if we take $q = 1$ and $r \geq 1$ in (2.32) and (2.33), then we get

$$\|(T_1, \dots, T_n)\|_{h,n,r} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|(T_1, \dots, T_n)\|_{h,n,r} \quad (2.39)$$

and

$$w_{h,n,r}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq n^{\frac{r-1}{r}} w_{h,n,r}(T_1, \dots, T_n) \quad (2.40)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

In particular, for $r = 2$ we get

$$\|(T_1, \dots, T_n)\|_{h,n,e} \leq \|(T_1, \dots, T_n)\|_{h,n,1} \leq \sqrt{n} \|(T_1, \dots, T_n)\|_{h,n,e} \quad (2.41)$$

and

$$w_{n,e}(T_1, \dots, T_n) \leq w_{h,n,1}(T_1, \dots, T_n) \leq \sqrt{n} w_{n,e}(T_1, \dots, T_n) \quad (2.42)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

We have:

PROPOSITION 10. For any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\|(T_1, \dots, T_n)\|_{h,n,q} \geq \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\| \quad (2.43)$$

and

$$w_{h,n,q}(T_1, \dots, T_n) \geq \frac{1}{n^{1/p}} w \left(\sum_{j=1}^n T_j \right). \quad (2.44)$$

Proof. Let $\lambda_j = \frac{1}{n^{1/p}}$ for $j \in \{1, \dots, n\}$, then $\sum_{j=1}^n |\lambda_j|^p = 1$. Therefore by (1.8) we get

$$\|(T_1, \dots, T_n)\|_{h,n,q} = \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^n \lambda_j T_j \right\| \geq \left\| \sum_{j=1}^n \frac{1}{n^{1/p}} T_j \right\| = \frac{1}{n^{1/p}} \left\| \sum_{j=1}^n T_j \right\|.$$

The inequality (2.44) follows in a similar way. ■

We can also introduce the following norms for $(T_1, \dots, T_n) \in B^{(n)}(H)$,

$$\|(T_1, \dots, T_n)\|_{s,n,p} := \sup_{\|x\|=1} \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p}$$

where $p \geq 1$ and

$$\|(T_1, \dots, T_n)\|_{s,n,\infty} := \sup_{\|x\|=1} \left(\max_{j \in \{1, \dots, n\}} \|T_j x\| \right) = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\}.$$

The triangle inequality $\|\cdot\|_{s,n,q}$ follows by Minkowski inequality, while the other properties of the norm are obvious.

PROPOSITION 11. Let $(T_1, \dots, T_n) \in B^{(n)}(H)$.

(i) We have for $p \geq 1$, that

$$\begin{aligned} \|(T_1, \dots, T_n)\|_{h,n,p} &\leq \|(T_1, \dots, T_n)\|_{s,n,p} \\ &\leq \|(T_1, \dots, T_n)\|_{n,p}, \end{aligned} \quad (2.45)$$

(ii) For $p \geq 2$ we also have

$$\|(T_1, \dots, T_n)\|_{s,n,p} = \left[w_{h,n,p/2}(|T_1|^2, \dots, |T_n|^2) \right]^{1/2}, \quad (2.46)$$

where the absolute value $|T|$ is defined by $|T| := (T^*T)^{1/2}$.

Proof. (i) We have for $p \geq 2$ and $x, y \in H$ with $\|x\| = \|y\| = 1$, that

$$\begin{aligned} |\langle T_j x, y \rangle|^p &\leq \|T_j x\|^p \|y\|^p \\ &= \|T_j x\|^p \leq \|T_j\|^p \|x\|^p = \|T_j\|^p \end{aligned}$$

for $j \in \{1, \dots, n\}$.

This implies

$$\sum_{j=1}^n |\langle T_j x, y \rangle|^p \leq \sum_{j=1}^n \|T_j x\|^p \leq \sum_{j=1}^n \|T_j\|^p,$$

namely

$$\left(\sum_{j=1}^n |\langle T_j x, y \rangle|^p \right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j\|^p \right)^{1/p}, \quad (2.47)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (2.47), we get the desired result (2.45).

(ii) We have

$$\begin{aligned}
& \|(T_1, \dots, T_n)\|_{s,n,p} \\
&= \sup_{\|x\|=1} \left(\sum_{j=1}^n \|T_j x\|^p \right)^{1/p} = \sup_{\|x\|=1} \left(\sum_{j=1}^n (\|T_j x\|^2)^{p/2} \right)^{1/p} \\
&= \sup_{\|x\|=1} \left(\sum_{j=1}^n \langle T_j x, T_j x \rangle^{p/2} \right)^{1/p} = \sup_{\|x\|=1} \left(\sum_{j=1}^n \langle T_j^* T_j x, x \rangle^{p/2} \right)^{1/p} \\
&= \sup_{\|x\|=1} \left(\sum_{j=1}^n \langle |T_j|^2 x, x \rangle^{p/2} \right)^{1/p} = \left[\sup_{\|x\|=1} \left(\sum_{j=1}^n \langle |T_j|^2 x, x \rangle^{p/2} \right)^{1/(p/2)} \right]^{1/2} \\
&= \left[w_{h,n,p/2}(|T_1|^2, \dots, |T_n|^2) \right]^{1/2},
\end{aligned}$$

which proves the equality (2.46). ■

3. SOME REVERSE INEQUALITIES

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [2] (see also [3, Theorem 5.14]):

LEMMA 12. Let $a, A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be two sequences of real numbers with the property that:

$$ay_j \leq z_j \leq Ay_j \quad \text{for each } j \in \{1, \dots, n\}. \quad (3.1)$$

Then for any $\mathbf{w} = (w_1, \dots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^n w_j z_j^2 \sum_{j=1}^n w_j y_j^2 - \left(\sum_{j=1}^n w_j z_j y_j \right)^2 \leq \frac{1}{4} (A - a)^2 \left(\sum_{j=1}^n w_j y_j^2 \right)^2. \quad (3.2)$$

The constant $\frac{1}{4}$ is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [15]) the following counterparts of (CBS)-inequality (see also [3, Theorem 5.20 & 5.21]):

LEMMA 13. Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists a, A, b, B with the property that:

$$0 \leq a \leq a_j \leq A \quad \text{and} \quad 0 < b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\}, \quad (3.3)$$

then we have the inequality

$$\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 - \left(\sum_{j=1}^n a_j b_j \right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^n a_j b_j \sum_{j=1}^n b_j^2. \quad (3.4)$$

and

LEMMA 14. Assume that \mathbf{a}, \mathbf{b} are nonnegative sequences and there exists γ, Γ with the property that

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.5)$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^n b_j^2. \quad (3.6)$$

We have:

THEOREM 15. Let $(T_1, \dots, T_n) \in B^{(n)}(H)$.

(i) We have

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \\ &\leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2 \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} 0 &\leq w_{n,e}^2(T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \dots, T_n) \\ &\leq \frac{1}{4} n \|(T_1, \dots, T_n)\|_{n,\infty}^2. \end{aligned} \quad (3.8)$$

(ii) We have

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \dots, T_n)\|_{h,n,1}^2 \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} \|(T_1, \dots, T_n)\|_{h,n,1} \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} 0 &\leq w_{n,e}^2(T_1, \dots, T_n) - \frac{1}{n} w_{h,n,1}^2(T_1, \dots, T_n) \\ &\leq \|(T_1, \dots, T_n)\|_{n,\infty} w_{h,n,1}(T_1, \dots, T_n). \end{aligned} \quad (3.10)$$

(iii) We have

$$\begin{aligned} 0 &\leq \|(T_1, \dots, T_n)\|_{h,n,e} - \frac{1}{\sqrt{n}} \|(T_1, \dots, T_n)\|_{h,n,1} \\ &\leq \frac{1}{4} \sqrt{n} \|(T_1, \dots, T_n)\|_{n,\infty} \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} 0 &\leq w_{n,e}(T_1, \dots, T_n) - \frac{1}{\sqrt{n}} w_{h,n,1}(T_1, \dots, T_n) \\ &\leq \frac{1}{4} \sqrt{n} \|(T_1, \dots, T_n)\|_{n,\infty}. \end{aligned} \quad (3.12)$$

Proof. (i) Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and put

$$R = \max_{j \in \{1, \dots, n\}} \{\|T_j\|\} = \|(T_1, \dots, T_n)\|_{n,\infty}.$$

If $x, y \in H$, with $\|x\| = \|y\| = 1$ then $|\langle T_j x, y \rangle| \leq \|T_j x\| \leq \|T_j\| \leq R$ for any $j \in \{1, \dots, n\}$.

If we write the inequality (3.2) for $z_j = |\langle T_j x, y \rangle|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^n |\langle T_j x, y \rangle|^2 - \left(\sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 + \frac{1}{4} n R^2 \quad (3.13)$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |\langle T_j x, x \rangle| \right)^2 + \frac{1}{4} n R^2 \quad (3.14)$$

for any $x \in H$, with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.13) and $\|x\| = 1$ in (3.14), then we get (3.7) and (3.8).

(ii) Let $(T_1, \dots, T_n) \in B^{(n)}(H)$. If we write the inequality (3.4) for $a_j = |\langle T_j x, y \rangle|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^n |\langle T_j x, y \rangle|^2 - \left(\sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 \leq n R \sum_{j=1}^n |\langle T_j x, y \rangle|,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |\langle T_j x, y \rangle| \right)^2 + R \sum_{j=1}^n |\langle T_j x, y \rangle|, \quad (3.15)$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |\langle T_j x, x \rangle| \right)^2 + R \sum_{j=1}^n |\langle T_j x, x \rangle|, \quad (3.16)$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.15) and $\|x\| = 1$ in (3.16), then we get (3.9) and (3.10).

(iii) If we write the inequality (3.6) for $a_j = |\langle T_j x, y \rangle|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \leq \left(n \sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n |\langle T_j x, y \rangle| \leq \frac{1}{4} n R,$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\left(\sum_{j=1}^n |\langle T_j x, y \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |\langle T_j x, y \rangle| + \frac{1}{4} \sqrt{n} R, \quad (3.17)$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |\langle T_j x, x \rangle| + \frac{1}{4} \sqrt{n} R, \quad (3.18)$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.17) and $\|x\| = 1$ in (3.18), then we get (3.11) and (3.12). ■

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If $\gamma, \Gamma \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$ with the property that

$$\begin{aligned} 0 &\leq \operatorname{Re} [(\Gamma - \alpha_j) (\overline{\alpha_j} - \overline{\gamma})] \\ &= (\operatorname{Re} \Gamma - \operatorname{Re} \alpha_j) (\operatorname{Re} \alpha_j - \operatorname{Re} \gamma) + (\operatorname{Im} \Gamma - \operatorname{Im} \alpha_j) (\operatorname{Im} \alpha_j - \operatorname{Im} \gamma) \end{aligned} \quad (3.19)$$

or, equivalently,

$$\left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \quad (3.20)$$

for each $j \in \{1, \dots, n\}$, then (see for instance [4, p. 9])

$$n \sum_{j=1}^n |\alpha_j|^2 - \left| \sum_{j=1}^n \alpha_j \right|^2 \leq \frac{1}{4} n^2 |\Gamma - \gamma|^2. \quad (3.21)$$

In addition, if $\operatorname{Re} (\Gamma \overline{\gamma}) > 0$, then (see for example [4, p. 26]):

$$\begin{aligned} n \sum_{j=1}^n |\alpha_j|^2 &\leq \frac{1}{4} \frac{\left\{ \operatorname{Re} \left[(\overline{\Gamma} + \overline{\gamma}) \sum_{j=1}^n \alpha_j \right] \right\}^2}{\operatorname{Re} (\Gamma \overline{\gamma})} (\Gamma \overline{\gamma}) \\ &\leq \frac{1}{4} \frac{|\Gamma + \gamma|^2}{\operatorname{Re} (\Gamma \overline{\gamma})} \left| \sum_{j=1}^n \alpha_j \right|^2. \end{aligned} \quad (3.22)$$

Also, if $\Gamma \neq -\gamma$, then (see for instance [4, p. 32]):

$$\left(n \sum_{j=1}^n |\alpha_j|^2 \right)^{\frac{1}{2}} - \left| \sum_{j=1}^n \alpha_j \right| \leq \frac{1}{4} n \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \quad (3.23)$$

Finally, from [7] we can also state that

$$n \sum_{j=1}^n |\alpha_j|^2 - \left| \sum_{j=1}^n \alpha_j \right|^2 \leq n \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \sum_{j=1}^n \alpha_j \right|, \quad (3.24)$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$.

We notice that a simple sufficient condition for (3.19) to hold is that

$$\operatorname{Re} \Gamma \geq \operatorname{Re} \alpha_j \geq \operatorname{Re} \gamma \quad \text{and} \quad \operatorname{Im} \Gamma \geq \operatorname{Im} \alpha_j \geq \operatorname{Im} \gamma \quad (3.25)$$

for each $j \in \{1, \dots, n\}$.

THEOREM 16. *Let $(T_1, \dots, T_n) \in B^{(n)}(H)$ and $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \gamma$. Assume that*

$$w\left(T_j - \frac{\gamma + \Gamma}{2} I\right) \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.26)$$

(i) *We have*

$$w_{h,n,e}^2(T_1, \dots, T_n) \leq \frac{1}{n} w^2\left(\sum_{j=1}^n T_j\right) + \frac{1}{4} n |\Gamma - \gamma|^2. \quad (3.27)$$

(ii) *If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then*

$$w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{2\sqrt{n}} \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} w\left(\sum_{j=1}^n T_j\right) \quad (3.28)$$

and

$$\begin{aligned} & w_{h,n,e}^2(T_1, \dots, T_n) \\ & \leq \left[\frac{1}{n} w\left(\sum_{j=1}^n T_j\right) + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \right] \cdot w\left(\sum_{j=1}^n T_j\right). \end{aligned} \quad (3.29)$$

(iii) If $\Gamma \neq -\gamma$, then

$$w_{h,n,e}(T_1, \dots, T_n) \leq \frac{1}{\sqrt{n}} \left(w \left(\sum_{j=1}^n T_j \right) + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right). \quad (3.30)$$

Proof. Let $x \in H$ with $\|x\| = 1$ and $(T_1, \dots, T_n) \in B^{(n)}(H)$ with the property (3.26). By taking $\alpha_j = \langle T_j x, x \rangle$ we have

$$\begin{aligned} \left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| &= \left| \langle T_j x, x \rangle - \frac{\gamma + \Gamma}{2} \langle x, x \rangle \right| = \left| \left\langle \left(T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right\rangle \right| \\ &\leq \sup_{\|x\|=1} \left| \left\langle \left(T_j - \frac{\gamma + \Gamma}{2} I \right) x, x \right\rangle \right| \\ &= w \left(T_j - \frac{\gamma + \Gamma}{2} I \right) \leq \frac{1}{2} |\Gamma - \gamma| \end{aligned}$$

for any $j \in \{1, \dots, n\}$.

(i) By using the inequality (3.21), we have

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, x \rangle|^2 &\leq \frac{1}{n} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \end{aligned} \quad (3.31)$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in (3.31) we get

$$\begin{aligned} \sup_{\|x\|=1} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right) &\leq \frac{1}{n} \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} w^2 \left(\sum_{j=1}^n T_j \right) + \frac{1}{4} n |\Gamma - \gamma|^2, \end{aligned}$$

which proves (3.27).

(ii) If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then by (3.22) we have for $\alpha_j = \langle T_j x, x \rangle$, $j \in \{1, \dots, n\}$ that

$$\begin{aligned} \sum_{j=1}^n |\langle T_j x, x \rangle|^2 &\leq \frac{1}{4n} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 \\ &= \frac{1}{4n} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 \end{aligned} \quad (3.32)$$

for any $x \in H$ with $\|x\| = 1$.

On taking the supremum over $\|x\| = 1$ in (3.32) we get (3.32).

Also, by (3.24) we get

$$\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \leq \frac{1}{n} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|,$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we have

$$\begin{aligned} &\sup_{\|x\|=1} \sum_{j=1}^n |\langle T_j x, x \rangle|^2 \\ &\leq \sup_{\|x\|=1} \left[\frac{1}{n} \left| \sum_{j=1}^n \langle T_j x, x \rangle \right|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \sum_{j=1}^n \langle T_j x, x \rangle \right| \right] \\ &\leq \frac{1}{n} \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right|^2 + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \sup_{\|x\|=1} \left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right| \\ &= \frac{1}{n} w^2 \left(\sum_{j=1}^n T_j \right) + \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] w \left(\sum_{j=1}^n T_j \right), \end{aligned}$$

which proves (3.29).

(iii) By the inequality (3.23) we have

$$\begin{aligned} \left(\sum_{j=1}^n |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{n}} \left(\left| \sum_{j=1}^n \langle T_j x, x \rangle \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \\ &= \frac{1}{\sqrt{n}} \left(\left| \left\langle \sum_{j=1}^n T_j x, x \right\rangle \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\|=1$ in this inequality, we get (3.30). ■

Remark 17. By the use of the elementary inequality $w(T) \leq \|T\|$ that holds for any $T \in B(H)$, a sufficient condition for (3.26) to hold is that

$$\left\| T_j - \frac{\gamma + \Gamma}{2} \right\| \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.33)$$

4. INEQUALITIES FOR δ_p AND ϑ_p NORMS

For $T \in B(H)$ and $p \geq 1$ we can consider the functionals

$$\delta_p(T) := \sup_{\|x\|=\|y\|=1} \left(|\langle Tx, y \rangle|^p + |\langle T^*x, y \rangle|^p \right)^{1/p} = \|(T, T^*)\|_{h,2,p} \quad (4.1)$$

and

$$\vartheta_p(T) := \sup_{\|x\|=1} \left(\|Tx\|^p + \|T^*x\|^p \right)^{1/p} = \|(T, T^*)\|_{s,2,p}. \quad (4.2)$$

It is easy to see that both δ_p and ϑ_p are norms on $B(H)$. The case $p = 2$ for the norm $\delta := \delta_2$ was considered and studied in [5].

Observe that, for any $T \in B(H)$ and $p \geq 1$, we have

$$\begin{aligned} w_{h,2,p}((T, T^*)) &= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^p + |\langle T^*x, x \rangle|^p \right)^{1/p} \\ &= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^p + |\langle Tx, x \rangle|^p \right)^{1/p} \\ &= 2^{1/p} \sup_{\|x\|=1} |\langle Tx, x \rangle| = 2^{1/p} w(T). \end{aligned} \quad (4.3)$$

Using the inequality (1.13) we have

$$2^{1/p} w(T) \leq \delta_p(T) \leq 2^{1+1/p} w(T) \quad (4.4)$$

for any $T \in B(H)$ and $p \geq 1$.

For $p = 2$, we get

$$\sqrt{2}w(T) \leq \delta(T) \leq \sqrt{8}w(T) \quad (4.5)$$

while for $p = 1$ we get

$$2w(T) \leq \delta_1(T) \leq 4w(T) \quad (4.6)$$

for any $T \in B(H)$.

We have for any $T \in B(H)$ and $p \geq 1$ that

$$\|(T, T^*)\|_{2,p} = (\|T\|^p + \|T^*\|^p)^{1/p} = 2^{1/p}\|T\|$$

and by (2.25) we get

$$\|T\| \leq \delta_p(T) \leq 2^{1/p}\|T\| \quad (4.7)$$

for any $T \in B(H)$ and $p \geq 1$.

For $p = 2$, we get

$$\|T\| \leq \delta(T) \leq \sqrt{2}\|T\| \quad (4.8)$$

while for $p = 1$ we get

$$\|T\| \leq \delta_1(T) \leq 2\|T\| \quad (4.9)$$

for any $T \in B(H)$.

From (2.32) we get for $r \geq q \geq 1$ that

$$\delta_r(T) \leq \delta_q(T) \leq 2^{\frac{r-q}{rq}} \delta_r(T) \quad (4.10)$$

for any $T \in B(H)$.

For any $T \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (2.43) we have

$$\delta_q(T) \geq \frac{1}{2^{1/p}}\|T + T^*\|. \quad (4.11)$$

In particular, for $p = q = 2$ we get

$$\delta(T) \geq \frac{\sqrt{2}}{2}\|T + T^*\|, \quad (4.12)$$

for any $T \in B(H)$.

By using the inequality (2.45) we get

$$\delta_p(T) \leq \vartheta_p(T) \leq 2^{1/p}\|T\| \quad (4.13)$$

for any $T \in B(H)$ and $p \geq 1$.

For $p = 1$ we get

$$\delta_1(T) \leq \vartheta_1(T) \leq 2\|T\| \quad (4.14)$$

for any $T \in B(H)$.

For $p \geq 2$, by employing the equality (2.46) we get

$$\vartheta_p(T) = \left[w_{h,2,p/2}(|T|^2, |T^*|^2) \right]^{1/2} = \left[2^{2/p} w(|T|^2) \right]^{1/2} = 2^{1/p} \|T\| \quad (4.15)$$

for any $T \in B(H)$.

On utilising (3.7), (3.9) and (3.11) we get

$$0 \leq \delta^2(T) - \frac{1}{2}\delta_1^2(T) \leq \frac{1}{2}\|T\|^2, \quad (4.16)$$

$$0 \leq \delta^2(T) - \frac{1}{2}\delta_1^2(T) \leq \|T\|\delta_1(T) \quad (4.17)$$

and

$$0 \leq \delta(T) - \frac{1}{\sqrt{2}}\delta_1(T) \leq \frac{\sqrt{2}}{4}\|T\| \quad (4.18)$$

for any $T \in B(H)$.

Observe, by (4.3) we have that

$$w_{h,2,e}((T, T^*)) = \sqrt{2}w(T),$$

for any $T \in B(H)$.

Assume that $T \in B(H)$ and $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \gamma$ such that

$$w\left(T - \frac{\gamma + \Gamma}{2}I\right), w\left(T^* - \frac{\gamma + \Gamma}{2}I\right) \leq \frac{1}{2}|\Gamma - \gamma|, \quad (4.19)$$

then by (3.27) we get

$$w^2(T) \leq \|\operatorname{Re}(T)\|^2 + \frac{1}{4}|\Gamma - \gamma|^2, \quad (4.20)$$

where $\operatorname{Re}(T) := \frac{T+T^*}{2}$.

If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then by (3.28) and (3.29)

$$w(T) \leq \frac{1}{2} \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \|\operatorname{Re}(T)\| \quad (4.21)$$

and

$$w^2(T) \leq \left[\|\operatorname{Re}(T)\| + \left[|\Gamma + \gamma| - 2\sqrt{(\Gamma\bar{\gamma})} \right] \right] \|\operatorname{Re}(T)\|. \quad (4.22)$$

If $\Gamma \neq -\gamma$, then by (3.30) we get

$$w(T) \leq \|\operatorname{Re}(T)\| + \frac{1}{8} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \quad (4.23)$$

Due to the fact that $w(A) = w(A^*)$ for any $A \in B(H)$, the condition (4.19) can be simplified as follows.

If m, M are real numbers with $M > m$ and if

$$w\left(T - \frac{m+M}{2}I\right) \leq \frac{1}{2}(M-m),$$

then

$$w^2(T) \leq \|\operatorname{Re}(T)\|^2 + \frac{1}{4}(M-m)^2. \quad (4.24)$$

If $m > 0$, then

$$w(T) \leq \frac{1}{2} \frac{m+M}{\sqrt{mM}} \|\operatorname{Re}(T)\| \quad (4.25)$$

and

$$w^2(T) \leq \left[\|\operatorname{Re}(T)\| + \left(\sqrt{M} - \sqrt{m} \right)^2 \right] \|\operatorname{Re}(T)\|. \quad (4.26)$$

If $M \neq -m$, then

$$w(T) \leq \|\operatorname{Re}(T)\| + \frac{1}{8} \frac{(M-m)^2}{m+M}. \quad (4.27)$$

For other numerical radius and norm inequalities, the interested reader may also consult [1] and [6] and compare these results. The details are not provided here.

5. INEQUALITIES FOR REAL NORMS

If X is a complex linear space, then the functional $\|\cdot\|$ is a real norm, if the homogeneity property in the definition of the norms is satisfied only for real numbers, namely we have

$$\|\alpha x\| = |\alpha| \|x\| \quad \text{for any } \alpha \in \mathbb{R} \text{ and } x \in X.$$

For instance if we consider the complex linear space of complex numbers \mathbb{C} then the functionals

$$\begin{aligned} |z|_p &:= (|\operatorname{Re}(z)|^p + |\operatorname{Im}(z)|^p)^{1/p}, \quad p \geq 1, \\ |z|_\infty &:= \max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\}, \quad p = \infty, \end{aligned}$$

are real norms on \mathbb{C} .

For $T \in B(H)$ we consider the Cartesian decomposition

$$T = \operatorname{Re}(T) + i \operatorname{Im}(T)$$

where the selfadjoint operators $\operatorname{Re}(T)$ and $\operatorname{Im}(T)$ are uniquely defined by

$$\operatorname{Re}(T) = \frac{T + T^*}{2} \quad \text{and} \quad \operatorname{Im}(T) = \frac{T - T^*}{2i}.$$

We can introduce the following functionals

$$\|T\|_{r,p} := (\|\operatorname{Re}(T)\|^p + \|\operatorname{Im}(T)\|^p)^{1/p}, \quad p \geq 1,$$

and

$$\|T\|_{r,\infty} := \max\{\|\operatorname{Re}(T)\|, \|\operatorname{Im}(T)\|\}, \quad p = \infty,$$

where $\|\cdot\|$ is the usual operator norm on $B(H)$. *The definition can be extended for any other norms on $B(H)$ or its subspaces.*

Using the properties of the norm $\|\cdot\|$ and the Minkowski's inequality

$$(|a+b|^p + |c+d|^p)^{1/p} \leq (|a|^p + |c|^p)^{1/p} + (|b|^p + |d|^p)^{1/p}$$

for $p \geq 1$ and $a, b, c, d \in \mathbb{C}$, we observe that $\|\cdot\|_{r,p}$, $p \in [1, \infty]$ is a real norm on $B(H)$.

For $p \geq 1$ and $T \in B$ we can introduce the following functionals

$$\begin{aligned} \eta_{r,p}(T) &:= \sup_{\|x\|=\|y\|=1} (|\operatorname{Re}\langle Tx, y \rangle|^p + |\operatorname{Im}\langle Tx, y \rangle|^p)^{1/p} \\ &= \sup_{\|x\|=\|y\|=1} (|\langle \operatorname{Re} T x, y \rangle|^p + |\langle \operatorname{Im} T x, y \rangle|^p)^{1/p} \\ &= \|(\operatorname{Re} T, \operatorname{Im} T)\|_{h,2,p}, \end{aligned}$$

$$\begin{aligned}
\theta_{r,p}(T) &:= \sup_{\|x\|=1} \left(|\operatorname{Re} \langle Tx, x \rangle|^p + |\operatorname{Im} \langle Tx, x \rangle|^p \right)^{1/p} \\
&= \sup_{\|x\|=1} \left(|\langle \operatorname{Re} T x, x \rangle|^p + |\langle \operatorname{Im} T x, x \rangle|^p \right)^{1/p} \\
&= w_{h,2,p}(\operatorname{Re} T, \operatorname{Im} T)
\end{aligned}$$

and

$$\kappa_{r,p}(T) := \sup_{\|x\|=1} \left(\|\operatorname{Re} T x\|^p + \|\operatorname{Im} T x\|^p \right)^{1/p} = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{s,2,p}.$$

The case $p = 2$ is of interest since for $T \in B(H)$ we have

$$\begin{aligned}
\eta_{r,2}(T) &:= \sup_{\|x\|=\|y\|=1} \left(|\operatorname{Re} \langle Tx, y \rangle|^2 + |\operatorname{Im} \langle Tx, y \rangle|^2 \right)^{1/2} \\
&= \sup_{\|x\|=\|y\|=1} |\langle Tx, y \rangle| = \|T\|, \\
\theta_{r,2}(T) &:= \sup_{\|x\|=1} \left(|\operatorname{Re} \langle Tx, x \rangle|^2 + |\operatorname{Im} \langle Tx, x \rangle|^2 \right)^{1/2} \\
&= \sup_{\|x\|=1} |\langle Tx, x \rangle| = w(T)
\end{aligned}$$

and

$$\begin{aligned}
\kappa_{r,2}(T) &:= \sup_{\|x\|=1} \left(\|\operatorname{Re} T x\|^2 + \|\operatorname{Im} T x\|^2 \right)^{1/2} \\
&= \sup_{\|x\|=1} \left(\left\langle (\operatorname{Re} T)^2 x, x \right\rangle + \left\langle (\operatorname{Im} T)^2 x, x \right\rangle \right)^{1/2} \\
&= \sup_{\|x\|=1} \left(\left\langle [(\operatorname{Re} T)^2 + (\operatorname{Im} T)^2] x, x \right\rangle \right)^{1/2} \\
&= \left\| (\operatorname{Re} T)^2 + (\operatorname{Im} T)^2 \right\|^{1/2} = \left\| \frac{|T|^2 + |T^*|^2}{2} \right\|^{1/2}.
\end{aligned}$$

For $p = \infty$ we have

$$\begin{aligned}\eta_{r,\infty}(T) &:= \sup_{\|x\|=\|y\|=1} \left(\max \{ |\operatorname{Re} \langle Tx, y \rangle|, |\operatorname{Im} \langle Tx, y \rangle| \} \right) \\ &= \max \left\{ \sup_{\|x\|=\|y\|=1} |\langle \operatorname{Re} T x, y \rangle|, \sup_{\|x\|=\|y\|=1} |\langle \operatorname{Im} T x, y \rangle| \right\} \\ &= \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \},\end{aligned}$$

and in a similar way

$$\theta_{r,\infty}(T) = \kappa_{r,\infty}(T) = \max \{ \|\operatorname{Re} T\|, \|\operatorname{Im} T\| \} = \|T\|_{r,\infty}.$$

The functionals $\eta_{r,p}$, $\theta_{r,p}$ and $\kappa_{r,p}$ with $p \in [1, \infty]$ are real norms on $B(H)$. We have

$$\begin{aligned}\eta_{r,p}(T) &= \sup_{\|x\|=\|y\|=1} \left(|\operatorname{Re} \langle Tx, y \rangle|^p + |\operatorname{Im} \langle Tx, y \rangle|^p \right)^{1/p} \\ &\leq \left(\sup_{\|x\|=\|y\|=1} |\operatorname{Re} \langle Tx, y \rangle|^p + \sup_{\|x\|=\|y\|=1} |\operatorname{Im} \langle Tx, y \rangle|^p \right)^{1/p} \\ &= (\|\operatorname{Re}(T)\|^p + \|\operatorname{Im}(T)\|^p)^{1/p} = \|T\|_{r,p}\end{aligned}$$

and

$$\begin{aligned}\|T\|_{r,\infty} &= \sup_{\|x\|=\|y\|=1} \left(\max \{ |\operatorname{Re} \langle Tx, y \rangle|, |\operatorname{Im} \langle Tx, y \rangle| \} \right) \\ &\leq \sup_{\|x\|=\|y\|=1} \left(|\operatorname{Re} \langle Tx, y \rangle|^p + |\operatorname{Im} \langle Tx, y \rangle|^p \right)^{1/p} = \eta_{r,p}(T)\end{aligned}$$

for any $p \geq 1$ and $T \in B(H)$.

In a similar way we have

$$\|T\|_{r,\infty} \leq \theta_{r,p}(T) \leq \|T\|_{r,p}$$

and

$$\|T\|_{r,\infty} \leq \kappa_{r,p}(T) \leq \|T\|_{r,p}$$

for any $p \geq 1$ and $T \in B(H)$.

If we write the inequality (1.13) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$ then we get

$$\theta_{r,p}(T) \leq \eta_{r,p}(T) \leq 2\theta_{r,p}(T) \quad (5.1)$$

for any $p \geq 1$ and $T \in B(H)$.

Using the inequalities (2.25) and (2.26) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$ then we get

$$\frac{1}{2^{1/p}} \|T\|_{r,p} \leq \eta_{r,p}(T) \leq \|T\|_{r,p} \quad (5.2)$$

and

$$\frac{1}{2^{1/p}} \|T\|_{r,p} \leq \theta_{r,p}(T) \leq \|T\|_{r,p} \quad (5.3)$$

for any $p \geq 1$ and $T \in B(H)$.

If we use the inequalities (2.32) and (2.33) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$ then we get for $t \geq p \geq 1$ that

$$\eta_{r,t}(T) \leq \eta_{r,p}(T) \leq 2^{\frac{t-p}{tp}} \eta_{r,t}(T) \quad (5.4)$$

and

$$\theta_{r,t}(T) \leq \theta_{r,p}(T) \leq 2^{\frac{t-p}{tp}} \theta_{r,t}(T) \quad (5.5)$$

for any $T \in B(H)$.

For $p = 1$ we have the functionals

$$\eta_{r,1}(T) = \sup_{\|x\|=\|y\|=1} (|\langle \operatorname{Re} Tx, y \rangle| + |\langle \operatorname{Im} Tx, y \rangle|) = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{h,2,1},$$

$$\theta_{r,1}(T) := \sup_{\|x\|=1} (|\langle \operatorname{Re} Tx, x \rangle| + |\langle \operatorname{Im} Tx, x \rangle|) = w_{h,2,1}(\operatorname{Re} T, \operatorname{Im} T)$$

and

$$\kappa_{r,1}(T) := \sup_{\|x\|=1} (\|\operatorname{Re} Tx\| + \|\operatorname{Im} Tx\|) = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{s,2,1}.$$

By utilising the inequalities (3.7), (3.9) and (3.11) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$, then

$$0 \leq \|T\|^2 - \frac{1}{2} \eta_{r,1}^2(T) \leq \frac{1}{2} (\max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\})^2, \quad (5.6)$$

$$0 \leq \|T\|^2 - \frac{1}{2} \eta_{r,1}^2(T) \leq \max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \eta_{r,1}(T) \quad (5.7)$$

and

$$0 \leq \|T\| - \frac{\sqrt{2}}{2} \eta_{r,1}(T) \leq \frac{\sqrt{2}}{4} \max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \quad (5.8)$$

for any $T \in B(H)$.

Also, by utilising the inequalities (3.8), (3.10) and (3.12) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$, then

$$0 \leq w^2(T) - \frac{1}{2} \theta_{r,1}^2(T) \leq \frac{1}{2} (\max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\})^2, \quad (5.9)$$

$$0 \leq w^2(T) - \frac{1}{2} \theta_{r,1}^2(T) \leq \max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \theta_{r,1}(T) \quad (5.10)$$

and

$$0 \leq w(T) - \frac{\sqrt{2}}{2} \theta_{r,1}(T) \leq \frac{\sqrt{2}}{4} \max \{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \quad (5.11)$$

for any $T \in B(H)$.

If m, M are real numbers with $M > m$ and if

$$\left\| \operatorname{Re} T - \frac{m+M}{2} I \right\|, \left\| \operatorname{Im} T - \frac{m+M}{2} I \right\| \leq \frac{1}{2} (M - m), \quad (5.12)$$

then by (3.27) we get

$$w^2(T) \leq \frac{1}{2} \|\operatorname{Re} T + \operatorname{Im} T\|^2 + \frac{1}{2} (M - m)^2. \quad (5.13)$$

If $m > 0$, then (3.28) and (3.29) we have

$$w(T) \leq \frac{1}{2\sqrt{2}} \frac{m+M}{\sqrt{mM}} \|\operatorname{Re} T + \operatorname{Im} T\| \quad (5.14)$$

and

$$w^2(T) \leq \left[\frac{1}{2} \|\operatorname{Re} T + \operatorname{Im} T\| + \left(\sqrt{M} - \sqrt{m} \right)^2 \right] \|\operatorname{Re} T + \operatorname{Im} T\|. \quad (5.15)$$

If $M \neq -m$, then by (3.30) we get

$$w(T) \leq \frac{1}{\sqrt{2}} \left(\|\operatorname{Re} T + \operatorname{Im} T\| + \frac{1}{4} \frac{(M - m)^2}{M + m} \right). \quad (5.16)$$

Finally, we observe that a simple sufficient condition for (5.12) to hold, is that

$$mI \leq \operatorname{Re} T, \quad \operatorname{Im} T \leq MI$$

in the operator order of $B(H)$.

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